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# Numerical Simulation of One Step Block Method for Treatment of Second Order Forced Motions in Mass-Spring Systems

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***Authors' contributions***

*This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.*

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## ABSTRACT

This paper discuss the numerical simulation of one step block method for treatment of second order forced motions in mass-spring systems of initial value problems. The one step block method has been developed with the introduction of off-mesh point at both grid and off- grid points using interpolation and collocation procedure to increase computational burden which may jeopardize the accuracy of the method in terms of error. The basic properties of the one step block method was established and numerical analysis shown that the one step block method was found to be consistent, convergent and zero-stable. The one step block method was simulated on three highly stiff mathematical problems to validate the accuracy of the block method without reduction, and obviously the results shown are more accurate over the existing method in literature.

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## 1. INTRODUCTION

Mathematicians has developed mathematical models to help them understanding the physical phenomena in real life problems. These models frequently lead to equations involving some derivatives of an unknown function of single or several variables, which are called differential equations. Differential equations have vast application in many fields such as engineering, medicine, economics, operation research, psychology and anthropology.

Ross [1] stated some of the problems that involved differential equations as

i. the problem arising from determining the projectile motion, satellite, rocket or planet,
ii. the problem of how to determine the charge or current in an electric circuit,
iii. the study of chemical reactions and
iv. the study of decomposition rate of radioactive substance or population growth rate.

These problems obey certain scientific laws that involve rates of change of one or more quantities. Mathematically, these rates of change can be expressed by derivatives. When the problems are converted to mathematical equations they will form differential equations.

There are two types of differential equation, namely Ordinary Differential Equation (ODE) and Partial Differential Equation (PDE). ODE is a differential equation that has single independent variable, while PDE is differential equation with two or more variables [1].

In this paper, a simulation of single-step block method for the determination of motions on weights in a mass-spring systems of equation in the form of second order initial value problems of ordinary differential equation of the form

$$\frac{d^2x}{dt^2} = f\left(t, x, \frac{dx}{dt}\right), \; x(a) = y_0, \; \frac{dx}{dt}(b) = y_1, t \in [a,b] \tag{1.1}$$

shall be proposed.

Differential equations in the form of (1.1) play an important role in modeling virtually every physical or biological process because such equations occur in connection with numerous problems that are encountered in various aspects of our everyday life Abdelrahim & Omar [2] and Moaddy et al. [3]. Some scholars such as Omar & Kubuye, [4], Sunday, et al. [5], Ukpebor, Omole & Adoghe [6], Emmanuel, Ibenu, & Ezenweke [7], Skwame, Sabo & Mathew [8] and Sabo, Althemai & Hamadina [9] developed block method for simulation of problem (1.1) without reduction. The block methods have the advantage of generating independent solutions at selected grid point without overlapping. They also possess the properties of Runge-Kutta method of being self-starting and do not require starting values.

## 2. THE DIFFERENTIAL EQUATION OF THE VIBRATIONS OF MASS-SPRING SYSTEMS

Consider the natural (unstretched) length of a coil spring $l$ and let *m* be *a* mass which is attached to the lower end of the spring so that it comes to rest in its equilibrium position $O$, this stretches the spring by an amount $e$, so that the stretched length is $l+e$. At the equilibrium position $O$, the mass *m* is acted upon by two forces i.e. the weight $mg$ acting vertically downwards and the spring force $ke$ acting vertically upwards. Thus, we have

$$mg = ke \tag{2.1}$$

Supposing *P* is the position of the mass (below equilibrium position) at any time *t* so that the distance from the equilibrium position *O* to the point *P* is given by $OP = x$. Then *x* may be positive, zero or negative according to whether the mass is below, at, or above its equilibrium position.

When the mass is situated at *P*, it is acted upon by the following forces. The forces tending to pull the mass downward are positive, while those pulling it vertically upward are negative.

i. $F_1 = mg$ acting in the vertically downward direction,
ii. Let $F_2$ be the restoring force of the spring. When the mass is at *P*, $F_2$ is acting in

theupward direction and so it is negative. By Hooke's law, we have,

$$F_2 = -k(x+e) \tag{2.2}$$

using (2.1) in (2.2), we get

$$F_2 = -(kx+mg) \tag{2.3}$$

iii. Let $F_3$ be the resisting force of the medium called damping force. It is known that for small velocities, $F_3$ is approximately proportional to the magnitude of the velocity. When the mass moving downward (at *P*, say), $F_3$ acts in the upward direction (opposite to that of the motion) and so $F_3$ is negative and is given by,

$$F_3 = -b\left(\frac{dx}{dt}\right) \tag{2.4}$$

iv. External impressed force $F_4 = F_4(t)$ acting in downward direction. By Newton's second law,

$$F = ma \tag{2.5}$$

Where $F = F_1 + F_2 + F_3 + F_4$ and $a = \frac{d^2x}{dt^2}$. Thus,

$$m\left(\frac{d^2x}{dt^2}\right) = mg - kx - mg - b\left(\frac{dx}{dt}\right) + F_4(t)$$

$$m\left(\frac{d^2x}{dt^2}\right) + b\left(\frac{dx}{dt}\right) + kx = F_4(t) \tag{2.6}$$

which is a differential equation for the motion of the mass on a spring and is of the form (1.1). If $b = 0$, the motion is called undamped otherwise it is called damped. If there are no external impressed forces, $F_4(t) = 0$ for all *t*, the motion is called free, otherwise it is called forced, Raisinghania [10] and Sunday, et al. [5].

## 3. MATHEMATICAL FORMULATION OF THE ONE STEP BLOCK METHOD

We consider the polynomial in the form,

$$y(x) = \sum_{j=0}^{p+q-1} a_j x^j \tag{3.1}$$

where $p$ and $q$ are number of distinct interpolation and collocation respectively.

We differentiating (3.1) twice, to obtain

$$\sum_{j=0}^{p+q-1} j(j-1) a_j x^{j-2} \tag{3.2}$$

Substituting (3.2) into (1.1) to give

$$\sum_{j=0}^{p+q-1} j(j-1) a_j x^{j-2} = f(x, y, y') \tag{3.3}$$

we interpolating (3.1) at point $x = x_{n+\frac{1}{8}}, x_{n+\frac{3}{8}}$ and collocating (3.3) at $x = x_n, x_{n+\frac{1}{8}}, x_{n+\frac{3}{8}}, x_{n+\frac{5}{8}}, x_{n+\frac{7}{8}}, x_{n+1}$ to give a system of equation written in matrix form as

$$\begin{pmatrix} 1 & x^1_{n+\frac{1}{8}} & x^2_{n+\frac{1}{8}} & x^3_{n+\frac{1}{8}} & x^4_{n+\frac{1}{8}} & x^5_{n+\frac{1}{8}} & x^6_{n+\frac{1}{8}} & x^7_{n+\frac{1}{8}} \\ 1 & x^1_{n+\frac{3}{8}} & x^2_{n+\frac{3}{8}} & x^3_{n+\frac{3}{8}} & x^4_{n+\frac{3}{8}} & x^5_{n+\frac{3}{8}} & x^6_{n+\frac{3}{8}} & x^7_{n+\frac{3}{8}} \\ 0 & 0 & 2 & 6x^1_n & 12x^2_n & 20x^3_n & 30x^4_n & 42x^4_n \\ 0 & 0 & 2 & 6x^1_{n+\frac{1}{8}} & 12x^2_{n+\frac{1}{8}} & 20x^3_{n+\frac{1}{8}} & 30x^4_{n+\frac{1}{8}} & 42x^4_{n+\frac{1}{8}} \\ 0 & 0 & 2 & 6x^1_{n+\frac{3}{8}} & 12x^2_{n+\frac{3}{8}} & 20x^3_{n+\frac{3}{8}} & 30x^4_{n+\frac{3}{8}} & 42x^4_{n+\frac{3}{8}} \\ 0 & 0 & 2 & 6x^1_{n+\frac{5}{8}} & 12x^2_{n+\frac{5}{8}} & 20x^3_{n+\frac{5}{8}} & 30x^4_{n+\frac{5}{8}} & 42x^4_{n+\frac{5}{8}} \\ 0 & 0 & 2 & 6x^1_{n+\frac{7}{8}} & 12x^2_{n+\frac{7}{8}} & 20x^3_{n+\frac{7}{8}} & 30x^4_{n+\frac{7}{8}} & 42x^4_{n+\frac{7}{8}} \\ 0 & 0 & 2 & 6x^1_{n+1} & 12x^2_{n+1} & 20x^3_{n+1} & 30x^4_{n+1} & 42x^4_{n+1} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{pmatrix} = \begin{pmatrix} y_{n+\frac{1}{8}} \\ y_{n+\frac{3}{8}} \\ f_n \\ f_{n+\frac{1}{8}} \\ f_{n+\frac{3}{8}} \\ f_{n+\frac{5}{8}} \\ f_{n+\frac{7}{8}} \\ f_{n+1} \end{pmatrix} \tag{3.4}$$

Solving for $a'_j s$ in the (3.4) and the resulting value of $a'_j s$ are substituted into (3.1) to yields a continuous implicit hybrid one step method of the form:

$$y(x)=\alpha_{\frac{1}{8}}(t)+\alpha_{\frac{3}{8}}(t)+\beta_0(t)+\beta_{\frac{1}{8}}(t)+\beta_{\frac{3}{8}}(t)+\beta_{\frac{5}{8}}(t)+\beta_{\frac{7}{8}}(t)+\beta_1(t) \tag{3.5}$$

where

$$\left.\begin{aligned}
\alpha_{\frac{1}{8}} &= \frac{3}{2}-4t \\
\alpha_{\frac{1}{4}} &= -\frac{1}{2}+4t \\
\beta_0 &= \frac{1}{2400}h^2-\frac{10597}{282240}t\,h^2+\frac{1}{2}t^2h^2-\frac{1513}{630}t^3h^2+\frac{192}{35}t^4h^2-\frac{3424}{525}t^5h^2+\frac{2048}{525}t^6h^2-\frac{2048}{2205}t^7h^2 \\
\beta_{\frac{1}{8}} &= \frac{7729}{430080}h^2-\frac{201253}{1128960}t\,h^2+\frac{10}{3}t^3h^2-\frac{673}{63}t^4h^2+\frac{1528}{105}t^5h^2-\frac{2944}{315}t^6h^2+\frac{1048}{441}t^7h^2 \\
\beta_{\frac{3}{8}} &= \frac{1957}{307200}h^2-\frac{6599}{161280}t\,h^2-\frac{14}{9}t^3h^2+\frac{137}{15}t^4h^2-\frac{1208}{75}t^5h^2+\frac{896}{75}t^6h^2-\frac{1024}{315}t^7h^2 \\
\beta_{\frac{5}{8}} &= -\frac{563}{307200}h^2+\frac{1447}{161280}t\,h^2+\frac{14}{15}t^3h^2-\frac{269}{45}t^4h^2+\frac{952}{75}t^5h^2-\frac{2432}{225}t^6h^2+\frac{1024}{315}t^7h^2 \\
\beta_{\frac{7}{8}} &= \frac{47}{61440}h^2-\frac{3707}{1128960}t\,h^2-\frac{10}{21}t^3h^2+\frac{199}{63}t^4h^2-\frac{152}{21}t^5h^2+\frac{2176}{315}t^6h^2-\frac{1024}{441}t^7h^2 \\
\beta_1 &= -\frac{17}{67200}h^2+\frac{293}{282240}t\,h^2+\frac{1}{6}t^3h^2-\frac{352}{315}t^4h^2+\frac{1376}{1575}t^5h^2-\frac{4096}{1575}t^6h^2+\frac{2048}{2205}t^7h^2
\end{aligned}\right\} \tag{3.6}$$

Evaluating (3.6) to obtain the continuous form as,

$$\begin{pmatrix} y_n \\ y_{n+\frac{5}{8}} \\ y_{n+\frac{7}{8}} \\ y_{n+1} \end{pmatrix}-\begin{pmatrix} y_{n+\frac{1}{8}} \\ y_{n+\frac{3}{8}} \end{pmatrix}\begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ -1 & 2 \\ -2 & 3 \\ -\frac{5}{2} & \frac{7}{2} \end{pmatrix}=h^2\begin{pmatrix} \frac{1}{2400} & \frac{7729}{430080} & \frac{1957}{307200} & -\frac{563}{307200} & \frac{47}{61440} & -\frac{17}{67200} \\ -\frac{3}{2800} & \frac{17}{2240} & \frac{119}{2400} & \frac{11}{1600} & -\frac{1}{1680} & \frac{1}{8400} \\ -\frac{17}{8400} & \frac{7}{480} & \frac{509}{4800} & \frac{19}{300} & \frac{43}{6720} & -\frac{1}{1200} \\ -\frac{37}{13440} & \frac{1621}{86016} & \frac{8137}{61440} & \frac{6017}{61440} & \frac{799}{28672} & -\frac{1}{1120} \end{pmatrix}\begin{pmatrix} f_n \\ f_{n+\frac{1}{8}} \\ f_{n+\frac{3}{8}} \\ f_{n+\frac{5}{8}} \\ f_{n+\frac{7}{8}} \\ f_{n+1} \end{pmatrix} \tag{3.7}$$

Differentiating (3.5) once, yields

$$y'(x)=\alpha'_{\frac{1}{8}}(t)+\alpha'_{\frac{3}{8}}(t)+\beta'_0(t)+\beta'_{\frac{1}{8}}(t)+\beta'_{\frac{3}{8}}(t)+\beta'_{\frac{5}{8}}(t)+\beta'_{\frac{7}{8}}(t)+\beta'_1(t) \tag{3.8}$$

where

$$\left.\begin{aligned}
\alpha'_{\frac{1}{8}} &= 4 \\
\alpha'_{\frac{3}{8}} &= 4 \\
\beta'_0 &= -\frac{10597}{282240}h^2+t\,h^2-\frac{1513}{210}t^2h^2+\frac{768}{35}t^3h^2-\frac{3424}{105}t^4h^2+\frac{4096}{175}t^5h^2-\frac{2048}{315}t^6h^2 \\
\beta'_{\frac{1}{8}} &= -\frac{201253}{1128960}h^2+10\,t^2h^2-\frac{2692}{63}t^3h^2+\frac{1528}{21}t^4h^2-\frac{5888}{105}t^5h^2+\frac{1024}{63}t^6h^2 \\
\beta'_{\frac{3}{8}} &= -\frac{6599}{161280}h^2-\frac{14}{3}t^2h^2+\frac{548}{15}t^3h^2-\frac{1208}{15}t^4h^2+\frac{1792}{25}t^5h^2-\frac{1024}{45}t^6h^2 \\
\beta'_{\frac{5}{8}} &= \frac{1447}{161280}h^2+\frac{14}{5}t^2h^2-\frac{1076}{45}t^3h^2+\frac{952}{15}t^4h^2-\frac{4864}{75}t^5h^2+\frac{1024}{45}t^6h^2 \\
\beta'_{\frac{7}{8}} &= -\frac{3707}{1128960}h^2-\frac{10}{7}t^2h^2+\frac{796}{63}t^3h^2-\frac{760}{21}t^4h^2+\frac{4352}{105}t^5h^2-\frac{1024}{63}t^6h^2 \\
\beta'_1 &= \frac{293}{282240}h^2+\frac{1}{2}t^2h^2-\frac{1408}{210}t^3h^2+\frac{1376}{105}t^4h^2-\frac{8192}{525}t^5h^2+\frac{2048}{315}t^6h^2
\end{aligned}\right\} \tag{3.9}$$

Evaluating (3.8) at all points, yields

$$\begin{pmatrix} hy'_n \\ hy'_{n+\frac{1}{8}} \\ hy'_{n+\frac{3}{8}} \\ hy'_{n+\frac{5}{8}} \\ hy'_{n+\frac{7}{8}} \\ hy'_{n+1} \end{pmatrix} - \begin{pmatrix} y_{n+\frac{1}{8}} \\ y_{n+\frac{3}{8}} \end{pmatrix} \begin{pmatrix} -4 & 4 \\ -4 & 4 \\ -4 & 4 \\ -4 & 4 \\ -4 & 4 \\ -4 & 4 \end{pmatrix} = h^2 \begin{pmatrix} \frac{57287}{967680} & \frac{57287}{967680} & \frac{57287}{967680} & \frac{57287}{967680} & \frac{57287}{967680} & \frac{57287}{967680} \\ \frac{923}{88200} & -\frac{1261}{14112} & -\frac{1513}{25200} & \frac{989}{50400} & -\frac{151}{17640} & \frac{253}{88200} \\ -\frac{701}{88200} & \frac{1513}{35280} & \frac{5297}{50400} & -\frac{131}{6300} & \frac{607}{70560} & -\frac{251}{8820} \\ -\frac{17}{17640} & \frac{1367}{70560} & \frac{1243}{5040} & \frac{1243}{10080} & -\frac{263}{17640} & \frac{73}{17640} \\ -\frac{589}{88200} & \frac{1289}{35280} & \frac{10393}{50400} & \frac{1801}{6300} & \frac{8279}{70560} & -\frac{1259}{88200} \\ -\frac{6841}{1411200} & \frac{35291}{1128960} & \frac{174877}{806400} & \frac{215107}{806400} & \frac{232837}{1128960} & \frac{47609}{1411200} \end{pmatrix} \begin{pmatrix} f_n \\ f_{n+\frac{1}{8}} \\ f_{n+\frac{3}{8}} \\ f_{n+\frac{5}{8}} \\ f_{n+\frac{7}{8}} \\ f_{n+1} \end{pmatrix} \quad (3.10)$$

solving (3.7) and (3.10) simultaneously, yields the explicit schemes as

$$\begin{pmatrix} y_{n+\frac{1}{8}} \\ y_{n+\frac{3}{8}} \\ y_{n+\frac{5}{8}} \\ y_{n+\frac{7}{8}} \\ y_{n+1} \end{pmatrix} - y_n - hy'_n \begin{pmatrix} \frac{1}{8} \\ \frac{3}{8} \\ \frac{5}{8} \\ \frac{7}{8} \\ 1 \end{pmatrix} = f_n \begin{pmatrix} \frac{48281}{11289600} \\ \frac{17139}{1254400} \\ \frac{9925}{4541584} \\ \frac{7007}{230400} \\ \frac{379}{11025} \end{pmatrix} h^2 + \begin{pmatrix} \frac{1217}{282240} & -\frac{4051}{3225600} & \frac{1147}{1612800} & -\frac{1601}{4515840} & \frac{1391}{11289600} \\ \frac{4905}{100352} & \frac{201}{22400} & -\frac{549}{358400} & \frac{117}{250880} & -\frac{171}{1254400} \\ \frac{45625}{451584} & \frac{8875}{129025} & \frac{25}{8064} & \frac{625}{903168} & -\frac{125}{451584} \\ \frac{14063}{92160} & \frac{31213}{230400} & \frac{26411}{460800} & \frac{49}{5760} & -\frac{343}{230400} \\ \frac{79}{441} & \frac{263}{1575} & \frac{143}{1575} & \frac{67}{2205} & -\frac{37}{22050} \end{pmatrix} \begin{pmatrix} f_{n+\frac{1}{8}} \\ f_{n+\frac{3}{8}} \\ f_{n+\frac{5}{8}} \\ f_{n+\frac{7}{8}} \\ f_{n+1} \end{pmatrix} \quad (3.11)$$

$$\begin{pmatrix} y'_{n+\frac{1}{8}} \\ y'_{n+\frac{3}{8}} \\ y'_{n+\frac{5}{8}} \\ y'_{n+\frac{7}{8}} \\ y'_{n+1} \end{pmatrix} - y'_n = f_n \begin{pmatrix} \frac{9679}{201600} \\ \frac{663}{22400} \\ \frac{295}{8064} \\ \frac{889}{28800} \\ \frac{103}{3150} \end{pmatrix} h + \begin{pmatrix} \frac{14339}{161280} & -\frac{2203}{115200} & \frac{409}{38400} & -\frac{851}{161280} & \frac{41}{22400} \\ \frac{3963}{17920} & \frac{1869}{12800} & -\frac{381}{12800} & \frac{213}{17920} & -\frac{87}{22400} \\ \frac{2125}{10752} & \frac{1325}{4608} & \frac{515}{4608} & -\frac{125}{10752} & \frac{25}{8064} \\ \frac{4949}{23040} & \frac{28469}{115200} & \frac{10633}{38400} & \frac{2779}{23040} & -\frac{49}{230400} \\ \frac{22}{105} & \frac{58}{225} & \frac{58}{225} & \frac{22}{105} & -\frac{103}{3150} \end{pmatrix} \begin{pmatrix} f_{n+\frac{1}{8}} \\ f_{n+\frac{3}{8}} \\ f_{n+\frac{5}{8}} \\ f_{n+\frac{7}{8}} \\ f_{n+1} \end{pmatrix} \quad (3.12)$$

## 4. NUMERICAL PROPERTIES OF THE BLOCK METHOD

The numerical properties of the block method, which includes the order, error constant, consistency, zero stability, convergence and region of absolute stability, shall be analyzed.

### 4.1 Order and Error Constant

Let the linear operator defined by $\ell[y(x);h]$, where,

$$\Delta\{y(x):h\} = A^{(0)}Y_m^{(i)} - \sum_{i=0}^{k} \frac{jh^{(i)}}{i!} y_n^{(i)} - h^{(3-1)}\left[d_i f(y_n) + b_i F(Y_m)\right], \quad (4.1)$$

Expanding $Y_m$ *and* $F(Y_m)$ in Taylor series and comparing the coefficients of $h$ gives

$$\Delta\{y(x):h\} = C_0 y(x) + C_1 y'(x) + \cdots + C_p h^p y^p(x) + C_{p+1} h^{p+1} y^{p+1}(x) + C_{p+2} h^{p+2} y^{p+2}(x) + \cdots \quad (4.2)$$

**Definition 4.1:** The linear operator $L$ and the associate block method are said to be of order $p$ if $C_0 = C_1 = \cdots = C_p = C_{p+1} = 0, \quad C_{p+2} \neq 0. \quad C_{p+2}$ is called the error constant and implies that the truncation error is given by $t_{n+k} = C_{p+2} h^{p+2} y^{p+3}(x) + 0h^{p+3}$

$$L\{y(x):h\} = C_0 y(x) + C_1 y'(x) + \cdots + C_p h^p y^p(x) + C_{p+1} h^{p+1} y^{p+1}(x) + C_{p+2} h^{p+2} y^{p+2}(x) + \cdots \tag{4.3}$$

$$\begin{bmatrix} \sum_{j=0}^{\infty} \frac{\left(\frac{1}{8}\right)^j}{j!} - y_n - \frac{1}{8} h y'_n - \frac{48281}{11289600} h y''_n - \sum_{j=0}^{\infty} \frac{h^{j+3}}{j!} y_n^{j+3} \left[ -\frac{1217}{282240}\left(\frac{1}{8}\right) + \frac{4051}{3225600}\left(\frac{3}{8}\right) + \frac{1147}{1612800}\left(\frac{5}{8}\right) + \frac{1601}{4515840}\left(\frac{7}{8}\right) - \frac{1391}{11289600}(1) \right] \\ \sum_{j=0}^{\infty} \frac{\left(\frac{3}{8}\right)^j}{j!} - y_n - \frac{3}{8} h y'_n - \frac{17139}{1254400} h y''_n - \sum_{j=0}^{\infty} \frac{h^{j+3}}{j!} y_n^{j+3} \left[ -\frac{4905}{100352}\left(\frac{1}{8}\right) - \frac{201}{22400}\left(\frac{3}{8}\right) + \frac{549}{358400}\left(\frac{5}{8}\right) - \frac{117}{250880}\left(\frac{7}{8}\right) + \frac{171}{1254400}(1) \right] \\ \sum_{j=0}^{\infty} \frac{\left(\frac{5}{8}\right)^j}{j!} - y_n - \frac{5}{8} h y'_n - \frac{9925}{451584} h y''_n - \sum_{j=0}^{\infty} \frac{h^{j+3}}{j!} y_n^{j+3} \left[ -\frac{45625}{451584}\left(\frac{1}{8}\right) - \frac{8875}{129024}\left(\frac{3}{8}\right) - \frac{25}{8064}\left(\frac{5}{8}\right) - \frac{625}{903168}\left(\frac{7}{8}\right) + \frac{125}{451584}(1) \right] \\ \sum_{j=0}^{\infty} \frac{\left(\frac{7}{8}\right)^j}{j!} - y_n - \frac{7}{8} h y'_n - \frac{7007}{230400} h y''_n - \sum_{j=0}^{\infty} \frac{h^{j+3}}{j!} y_n^{j+3} \left[ -\frac{14063}{92160}\left(\frac{1}{8}\right) - \frac{31213}{230400}\left(\frac{3}{8}\right) - \frac{26411}{460800}\left(\frac{5}{8}\right) - \frac{49}{5760}\left(\frac{7}{8}\right) + \frac{343}{230400}(1) \right] \\ \sum_{j=0}^{\infty} \frac{(1)^j}{j!} - y_n - h y'_n - \frac{375}{11025} h y''_n - \sum_{j=0}^{\infty} \frac{h^{j+3}}{j!} y_n^{j+3} \left[ -\frac{79}{441}\left(\frac{1}{8}\right) - \frac{263}{1575}\left(\frac{3}{8}\right) - \frac{143}{1575}\left(\frac{5}{8}\right) - \frac{67}{2205}\left(\frac{7}{8}\right) + \frac{37}{22050}(1) \right] \end{bmatrix} \tag{4.4}$$

Comparing the coefficient of $h$ in (4.4), according to Sabo, Althemai & Hamadina [9] the method is of order $p = 4$ and the error constant are given respectively by

$$C_{p+2} = \left[ -\frac{1217}{19818086400} \quad \frac{67}{734003200} \quad -\frac{25}{792723456} \quad \frac{343}{283155200} \quad \frac{37}{619315200} \right]^T$$

## 4.2 Consistency of the Method

**Definition 4.2:** A one step block method is said to be consistent if the order of the method order is greater than or equal to one. Therefore our method is consistent, Lambert [11].

## 4.3 Zero Stability of the Method

**Definition 4.3:** The one step block method is said to be zero-stable, if the roots of the first characteristics function $\pi(x)$ satisfied that $|x_z| \leq 1$, and if $|x_z| = 1$ satisfies $|q_s| \leq 1$ then, the multiplicity of $x_z$ must not greater than two. Now in order to find the zero-stable of the method, $\pi(x) = |xI - \hat{B}| = 0$, where $I$ and $\hat{B}$ are $5 \times 5$ identical matrix and the coefficient matrix of $y_n$ respectively. The matrix are sown below

$$\pi(x) = \left| xI - \hat{B}^{[2]_2} \right| = \left| x \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right| = \left| \begin{bmatrix} x & 0 & 0 & 0 & -1 \\ 0 & x & 0 & 0 & -1 \\ 0 & 0 & x & 0 & -1 \\ 0 & 0 & 0 & x & -1 \\ 0 & 0 & 0 & 0 & x-1 \end{bmatrix} \right| = x^4(x-1)$$

Thus, solving for $x$ in $x^5 - x^4$ gives $x = 0, 0, 0, 0, 1$. Hence the method is said to be zero stable, Fatunla [12].

## 4.4 Convergence of the Block Method

**Theorem 4.1:** the one step block method is said to be convergent if it is consistent and zero-stable. It is evident that our method is convergent, Henrici [13].

## 4.5 Absolute Stability Region of the Block Method

**Definition 4.4:** the region of absolute stability of the block method is been determine using the stability polynomial of the form

$$Aw - e_0 - e_1 - h^2 d - h^2 bw \tag{4.5}$$

The use of (4.5) is called the boundary locus method. Applying (4.5) we obtain the region of absolute stability according to Lambert [14] as

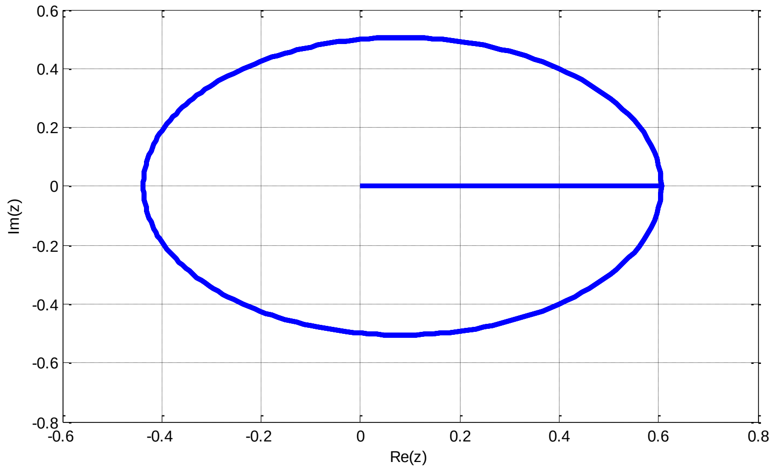


**Fig. 1. Shown the region of absolute stability of the block method**

## 5. MATHEMATICAL IMPLEMENTATION OF THE BLOCK METHOD

The accuracy and convergence of the block method will be studied using some highly stiff third order linear problems, tabular form and graphically shown.

### Mathematical Problem 5.1

### Real-life Problem (Mass Spring Motion)

A $128lb$ weight is attached to a spring having a spring constant of $64lb/ft$ . The weight is started in motion with no initial velocity by displacing it $6inches$ above the equilibrium position and by simultaneously applying to the weight an external force $F_4(t)=8\sin 4t$ . Assuming no air resistance, compute the subsequent motion of the weight at $t: 0.01 \le t \le 0.10$.

*Source:* [15]

Now, we model this problem into a mathematical equation of the form (1.1) using (2.6) and then apply our method to compute the motion on the weight attached to the spring. Here,

$$m=128,\ k=64,\ b=0,\ and\ F_4(t)=8\sin 4t$$

Thus, problem 5.1 boils down to

$$\frac{d^2x}{dt^2}+16x=2\sin 4t,\ x(0)=-\frac{1}{2},\ x'(0)=0 \quad (5.1)$$

with the exact solution of (5.1) is given by,

$$x(t)=-\frac{1}{2}\cos 4t+\frac{1}{16}\sin 4t-\frac{1}{4}t\cos 4t \quad (5.2)$$

**Table 1. Shown the Result for Mathematical Problem 5.1**

| t | Exact solution *(x)* | Computed solution *(x)* | Error in our method | Error in Skwame, Bakari & Sunday [8] |
|---|---|---|---|---|
| 0.01 | - 0.49959872021047678004 | - 0.49959872021047677994 | 1.0000e-19 | 1.6621e-09 |
| 0.02 | - 0.49839019330974949646 | - 0.49839019330974949605 | 4.1000e-19 | 1.1586e-08 |
| 0.03 | - 0.49636836974027966301 | - 0.49636836974027966210 | 9.1000e-19 | 2.9743e-08 |
| 0.04 | - 0.49352852660817937130 | - 0.49352852660817936964 | 1.6600e-18 | 5.6076e-08 |
| 0.05 | - 0.48986728796894500998 | - 0.48986728796894500736 | 2.6200e-18 | 9.0504e-08 |
| 0.06 | - 0.48538264289709933476 | - 0.48538264289709933096 | 3.8000e-18 | 1.3291e-07 |
| 0.07 | - 0.48007396129056685722 | - 0.48007396129056685202 | 5.2000e-18 | 1.8317e-07 |
| 0.08 | - 0.47394200736436189072 | - 0.47394200736436188387 | 6.8500e-18 | 2.4110e-07 |
| 0.09 | - 0.46698895079202783994 | - 0.46698895079202783119 | 8.7500e-18 | 3.0653e-07 |
| 1.00 | - 0.45921837545722401274 | - 0.45921837545722400189 | 1.0850e-17 | 3.7922e-07 |

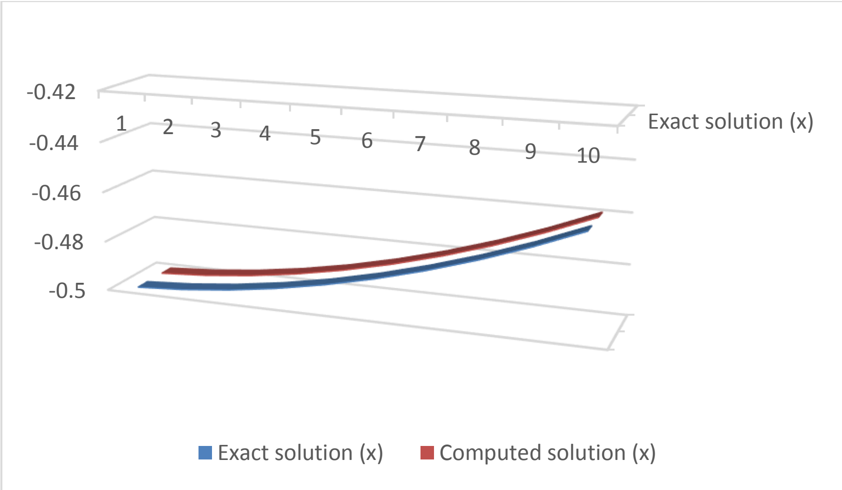


**Fig. 2. Shown the graphical solution showing the nature of mathematical problem 5.1**

### Mathematical Problem 5.2

### Real-life Problem (Dynamic Problems)

A $10\,kg$ mass is attached to a spring having a spring constant of $140\ N/M$ . The mass is started in motion from the equilibrium position with an initial velocity of 1 m/sec in the upward direction and with an applied external force $F(t)=5\sin t$ . Find the subsequent motion of the mass $(t:0.10\le t\le 1.00)$ if the force due to air resistance is $90\left(\frac{dx}{dt}\right)N$.

Applying the same procedure, where $m=10,\ k=140,\ a=90\ and\ F(t)=5\sin t$

Problem 5.2 reduces to

$$dsolver\left(\left\{\frac{d^2x}{dt^2}+9\frac{dx}{dt}+14x(t)=\frac{1}{2}\sin(t),\ x(0)=0,\ x'(0)=-1\right\}\right) \quad (5.3)$$

with the exact solution of (5.3) is given by,

$$x(t)=\frac{1}{500}\left(-90e^{-2t}+99e^{-7t}+13\sin t-9\cos t\right) \quad (5.4)$$

Source [Skwame, Bakari & Sunday [15] and Areo & Omojola [16].

### Mathematical Problem 5.3

### Highly Stiff Initial value problem

$$\frac{d^2x}{dt^2}-100x=0,\ x(0)=1,\ x'(0)=-10 \quad (5.5)$$

with the exact solution of (5.5) is given by,

$$x(t)=\exp(-10t) \quad (5.6)$$

Source [Kamoh, Abada & Soomiyol [17] and Areo & Omojola [16]]

**Table 2. Shown the Result for Mathematical Problem 5.2**

| t | Exact Result *(x)* | Computed Result *(x)* | Error in our Method | Error in Areo & Omojola [15] | Error in Skwame, Bakari & Sunday [8] |
|---|---|---|---|---|---|
| 0.1 | -0.06436205154552458248 | - 0.06436205175005231324 | 2.0453e-10 | 1.2744e-08 | 1.0647e-07 |
| 0.2 | -0.08430720522644774945 | - 0.08430720571129379133 | 4.8485e-10 | 3.0442e-08 | 1.1870e-06 |
| 0.3 | -0.08405225313390041905 | - 0.08405225379564348386 | 6.6174e-10 | 4.1501e-08 | 2.2635e-06 |
| 0.4 | -0.07529304213333374810 | - 0.07529304285982756949 | 7.2649e-10 | 4.5385e-08 | 2.8219e-06 |
| 0.5 | -0.06357063960355798563 | - 0.06357064031650558327 | 7.1295e-10 | 4.4298e-08 | 2.9539e-06 |
| 0.6 | -0.05142117069384508163 | - 0.05142117134934499406 | 6.5550e-10 | 4.0466e-08 | 2.8187e-06 |
| 0.7 | -0.03993052956438697070 | - 0.03993053014322618737 | 5.7884e-10 | 3.5475e-08 | 2.5466e-06 |
| 0.8 | -0.02949865862803573900 | - 0.02949865912611817948 | 4.9808e-10 | 3.0285e-08 | 2.2235 -06 |
| 0.9 | -0.02021269131259124546 | - 0.02021269173399231918 | 4.2140e-10 | 2.5408e-08 | 1.8991e-06 |
| 1.0 | -0.01202699425403169607 | - 0.01202699460659731142 | 3.5257e-10 | 2.1071e-08 | 1.5988e-06 |

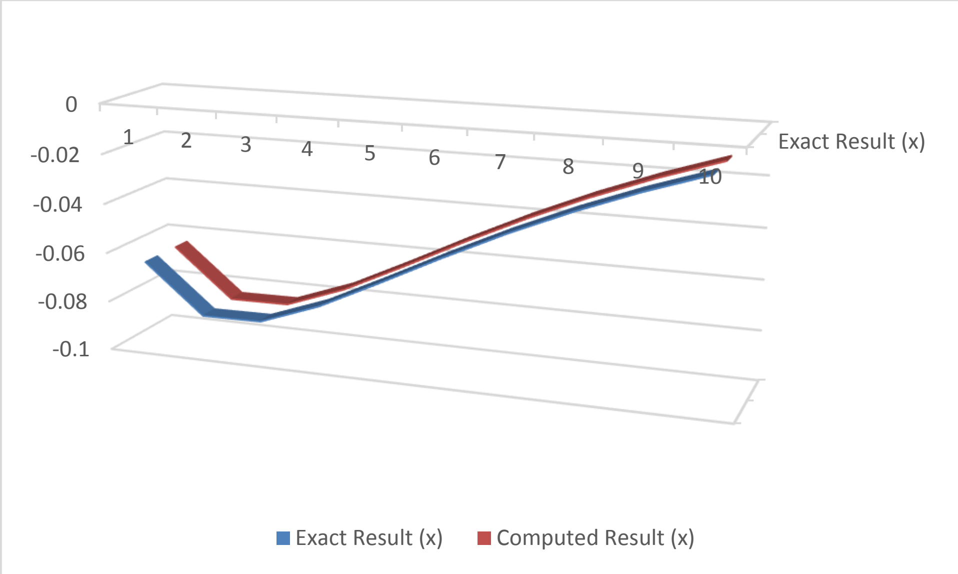


**Fig. 3. Shown the graphical solution showing the nature of mathematical problem 5.2**

**Table 3. Shown the Result for Mathematical Problem 5.3**

| t | **Exact Result *(x)*** | **Computed Result *(x)*** | **Error in our Method** | **Error in Kamoh, Abada & Soomiyol [17]** | **Error in Areo & Omojola [16]** |
|---|---|---|---|---|---|
| 0.1 | 0.90483741803595957316 | 0.90483741803595928948 | 2.8368e-16 | 6.6281e-11 | 1.8023e-12 |
| 0.2 | 0.81873075307798185867 | 0.81873075307798074703 | 1.1116e-15 | 1.6280e-10 | 7.0427e-12 |
| 0.3 | 0.74081822068171786607 | 0.74081822068171542544 | 2.4406e-15 | 2.5675e-10 | 1.5447e-11 |
| 0.4 | 0.67032004603563930074 | 0.67032004603563506342 | 4.2373e-15 | 3.4946e-10 | 2.6854e-11 |
| 0.5 | 0.60653065971263342360 | 0.60653065971262694607 | 6.4775e-15 | 4.7320e-10 | 4.1160e-11 |
| 0.6 | 0.54881163609402643263 | 0.54881163609401728714 | 9.1455e-15 | 7.0419e-10 | 5.7823e-11 |
| 0.7 | 0.49658530379140951470 | 0.49658530379139728133 | 1.2233e-14 | 9.6018e-10 | 7.7336e-11 |
| 0.8 | 0.44932896411722159143 | 0.44932896411720585059 | 1.5741e-14 | 1.2232e-09 | 9.9534e-11 |
| 0.9 | 0.40656965974059911188 | 0.40656965974057943717 | 1.9675e-14 | 1.4962e-09 | 1.2435e-10 |
| 1.0 | 0.36787944117144232160 | 0.36787944117141827282 | 2.4049e-14 | 1.8005e-09 | 1.5203e-10 |

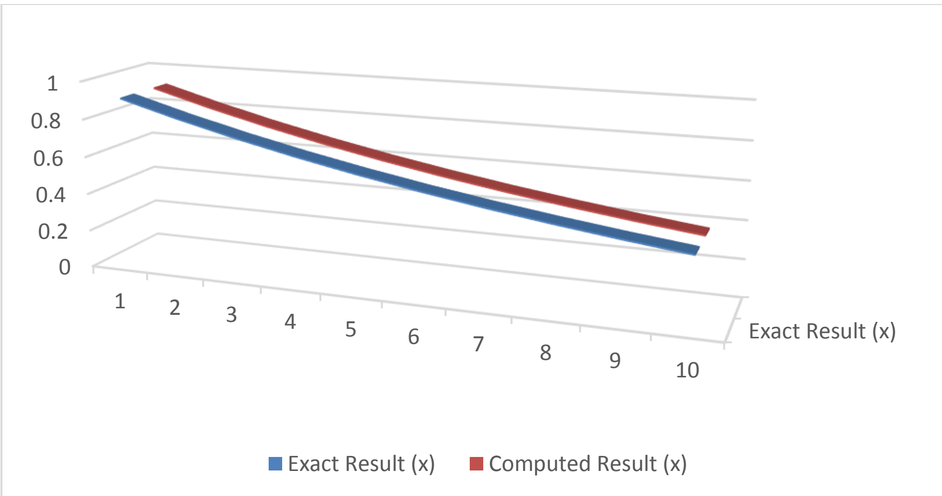


**Fig. 4. Shown the graphical solution showing the nature of mathematical problem 5.3**

## 6. SUMMARY AND CONCLUSION

Initially, initial value problems of higher order ordinary differential equations can be solved by first reducing the equations to an equivalent systems of first order ordinary differential equations. However, this approach will reduces the computational burden which may jeopardize the accuracy of the method. It was noticed that this reduction process has a lots of setbacks such as wastage of human efforts and computational burden which affects the accuracy of the method in terms of error. In order to overcoming the challenges, *the numerical simulation of one step block method for treatment of second order forced motions in mass-spring systems of initial value problems was proposed. The one step block method was developed with the introduction of off-mesh point at both grid and off- grid points using interpolation and collocation procedure to increase computational burden which may jeopardize the accuracy of the method in terms of error. The basic properties of the one step block method was established and numerical analysis shown that the one step block method was found to be consistent, convergent and zero-stable. The one step block method was simulated on some highly stiff mathematical problems to validate the accuracy of the block method without reduction, and graphically shown. Obviously the results shown in table 1 to 3 are more accurate over the existing method in literature.*

## COMPETING INTERESTS

Authors have declared that no competing interests exist.

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